Magnetic Deviation:
Comprehension, Compensation and Computation

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The Scottish mathematician and lawyer Archibald Smith first published in 1843 his equations for the magnetic deviation of a ship, or in other words, the error in the ship's compasses from permanent and induced magnetic fields in the iron of the ship itself. This effect had been noticed in mostly wooden ships for centuries, and broad attempts to minimize it were implemented. But the advent of ships with iron hulls and steam engines in the early 1800s created a real crisis. A mathematical formulation of the deviation for all compass courses and locations at sea was needed in order to understand and compensate for it, and Smith became the preeminent expert in this sphere of activity. With Capt. Frederick J. Evans he extended his mathematical treatment to detailed procedures for measuring the magnetic parameters for a ship, and he also invented graphical methods for quickly calculating the magnetic deviation for any ship's course once these parameters were found, constructions called dynamo-gonio-grams (force-angle diagrams), or *dygograms* for short.

Claude Joseph Vernet, *The Shipwreck*, 1772.

Today, radio navigational systems such as LORAN and GPS, and inertial navigation systems with ring and fiber-optic gyros, gyrocompasses and the like have reduced the use of a ship's compass to worst-case scenarios. But this triumph of mathematics and physics over the mysteries of magnetic deviation, entered into at a time when magnetic forces were barely understood and set against the backdrop of hundreds of shipwrecks and thousands of lost lives, is an enriching chapter in the history of science. This essay presents a brief sketch of the problem and the analysis and solutions that were developed to overcome it, followed by a discussion of Smith's graphical methods of computation.

The Sources of Compass Error

Assuming they are constructed well, compasses on ships fail to point to true (geographic) north due to two factors:

1. *Magnetic variation* (or *magnetic declination*), the angle between magnetic north and true north due to the local direction of the Earth's magnetic field, and

2. *Magnetic deviation*, the angle between the compass needle and magnetic north due to the presence of iron within the ship itself.

The algebraic sum of the magnetic variation and the magnetic deviation is known as the *compass error*. It is a very important thing to know.

Magnetic Variation

Magnetic variation has been known from voyages since the early 1400s at least. Certainly Columbus was distressed as he crossed the Atlantic to find that magnetic north and true north (from celestial sightings) drifted significantly, and in fact by 1542 it was known that an *agonic line*, where no difference between the two exists, runs through the Atlantic.

In 1581 Robert Norman published his conclusion, based on his records of magnetic variation, that the "point of respect" of a compass lies within the Earth rather than, say, a mountain of lodestone in the North as many supposed. This led to Dr. William Gilbert's construction of spherical lodestones to model the Earth and his proposal that it acts as a giant dipole magnet.

We now know that the locations of the Earth's magnetic poles are not coincident with the geographic poles—not even close, really—and they are always wandering around. Even then, the Earth's magnetic field is not a simple dipole, and geological masses can also affect the local magnetic field. Henry Gellibrand discovered in 1635 that there are also *secular* variations that change in time: slower ones due to changes in the Earth's magnetic field, and more sudden and temporary ones due to sunspot activity and magnetic storms in the ionosphere.

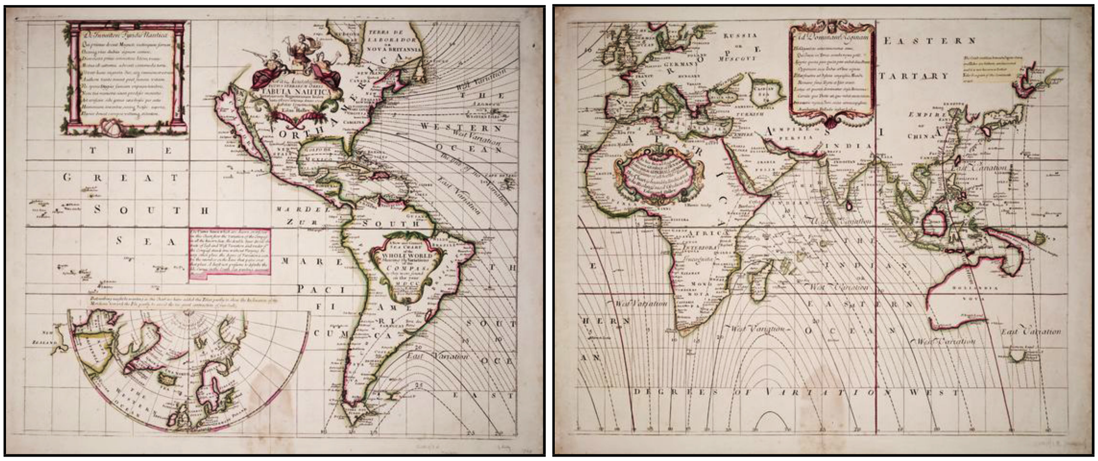
These magnetic variations are important, particularly on long ocean voyages. The mapping of these values led at one time to proposals to use on-board measurements of magnetic variation to determine the longitude of the ship; with sightings taken for the latitude this would provide a ship's location anywhere in the world. And knowing your location at sea was paramount. When the famous Longitude Prize was announced in England in 1714 (triggered by the loss of 2000 sailors and four ships of the Royal Navy off the Scilly Islands in 1707), the three main contenders for it were measurements of lunar parallax, clock time, and magnetic variation.

The first ocean voyages dedicated specifically to scientific research were those of Edmund Halley. In 1698 he commanded the ship *Paramour*, traveling the South Atlantic measuring the latitude, magnetic declination and longitude when possible.[1] Based on this and a subsequent voyage in 1699, Halley published in 1701 the world map below displaying the known magnetic variations,[2] information that was later ignored to his peril by Admiral Shovell, the commander of the fleet in the 1707 disaster. The bold line in this map that emerges from the southeastern U.S. and veers southward across the two halves and past West Africa is labeled the *The Line of No Variation*, the agonic line of his time. His "Curve Lines" of equal magnetic variation (today called

[1] The longitude was found by observing eclipses of the moons of Jupiter to retroactively determine universal time when compared against these events as recorded in England.

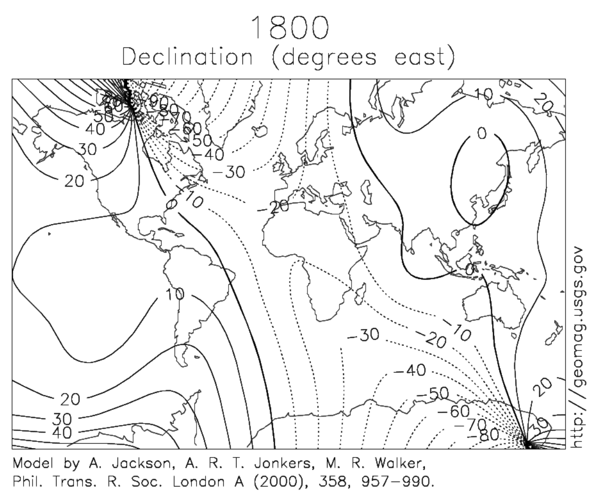
[2] In *A new and correct sea chart of the whole world showing the variations of the compass as they were found in the year M.D.CC.*, by Edmund Halley, found at *http://www.nmm.ac.uk/collections/explore/object.cfm?ID=G201%3A1%2F1&picture=1#content*

isogones) were first used by Christovao Bruno in the 1620s, but this is the earliest surviving example of such a map. It doesn't seem much of a stretch to me to imagine that this might have influenced Faraday in his concept of magnetic field lines 150 years later. Captain James Cook used a copy of Halley's map on his voyages around the world.



Edmund Halley's 1701 map of the known magnetic variations of the world.

As Halley noted, these maps must be regularly updated, and ships studiously logged the differences between their compass heading and true north obtained from astronomical sightings. A map of the world's magnetic variation in 1800 is shown to the right.[3] You can see that the bold agonic line had moved in that span of 100 years. The variation off the coast of England, as another example, increased to $-30°$ from less than $-10°$. Ultimately Halley modeled the Earth's magnetic field as four magnetic poles, two in the crust and two in an interior rotating ball, none with symmetry. It's a shame that Halley only infrequently measured the dip, or inclination, of the Earth's magnetic field, thinking it not nearly as important as the horizontal force.



Magnetic Deviation

There is an additional effect on the compass needle that took much longer to appreciate and even longer to understand. This **magnetic deviation** is due to the iron in a ship. Even the small amount of iron in wooden ships had an impact, although it was often masked by shoddy compass construction. The first notice in print of this effect was by Joao de Castro of Portugal in 1538, in which he identified "the proximity of artillery pieces, anchors and other iron" as the source. As better compass designs appeared, a difference in compass readings with their placement on the same ship became more apparent. Captains John Smith and James Cook warned about iron nails in the compass box or iron in steerage, and on Cook's second circumnavigation William Wales found that changes in the ship's course changed their measurements of magnetic variation by as much as 7°.

[3] From *http://geomag.usgs.gov/movies/* using US Geological Survey models.

Captain Matthew Flinders (1774-1815) spent years in the very early 1800s on voyages to investigate these effects, discovering that in addition to the horizontal magnetic field of the Earth, the inclination (or dip) of the field contributes to the magnetic deviation as well, or in other words, that both the vertical and horizontal components of the Earth's magnetic field affect a compass. He eventually discovered that an iron bar placed vertically near the compass helped overcome the magnetic deviation, and this **Flinder's bar** is still used today in ships' binnacles. Dr. William Scoresby later isolated the *soft iron* in the ship as being magnetized by the Earth's magnetic field and thereby affecting the compass.

The effects became much more pronounced after 1822 when construction of ships with iron hulls and steam engines commenced. Here it was discovered that *hard iron*, which becomes magnetic when pounded as during the construction of a ship, turns the ship into a permanent, multi-pole magnet. And this magnet changes slowly under the pounding of waves or vibrations from engines, or suddenly through collisions. George Airy, Royal Astronomer of Greenwich, initiated the procedure of *swinging* a ship to measure its deviation, and then to correct it with permanent magnets and chains of soft iron in the binnacles, but his corrective actions did not work worldwide, particularly south of the magnetic equator where the dip reversed sign or under the changing conditions mentioned above. Public disputes occurred between Airy and Scoresby over these magnetic compensation methods for ships [see Smith, 1869], with a third front opening from those such as F. J. Evans who preferred to simply subtract tabulated values of magnetic deviation at the ship locations.

The Mathematical Description of Magnetic Deviation

Beginning in 1843, Archibald Smith (1813-1872), a warm man "behind a reserve which is perhaps incident to engrossing thought" [Thomson], derived and developed his set of equations for magnetic deviation. They are expressed in terms of the ship's magnetic or compass course, the horizontal component and dip of the Earth's magnetic field at a given location, and magnetic parameters unique to a given ship. For narrative flow the essence of the derivation is relegated to the Appendix of this essay, but the two principal results are

$$\tan\delta = \frac{\mathbf{A} + \mathbf{B}\sin\xi + \mathbf{C}\cos\xi + \mathbf{D}\sin 2\xi + \mathbf{E}\cos 2\xi}{1 + \mathbf{B}\cos\xi - \mathbf{C}\sin\xi + \mathbf{D}\cos 2\xi - \mathbf{E}\sin 2\xi} \tag{1}$$

and

$$\delta = A + B\sin\xi' + C\cos\xi' + D\sin 2\xi' + E\cos 2\xi' \tag{2}$$

where the ship's head is pointing at an angle ξ from magnetic north at a given location (the *magnetic course*), and at an angle ξ' as shown on the compass (the *compass course*). The magnetic deviation δ of the compass needle from magnetic north due to the ship is then $\xi - \xi'$. The first equation above provides the deviation using exact coefficients **A**, **B**, **C**, **D** and **E** but is expressed in terms of the non-observable magnetic course, while the second equation is expressed in terms of the observable compass course but uses inexact coefficients A, B, C, D and E. Therefore, the second (inexact) equation is most useful on-ship, while the first (exact) one is more useful in characterizing the ship. The relationships of the terms in the second equation to those in the first are:

$$\delta = A + B\sin\xi' + C\cos\xi' + D\sin 2\xi' + E\cos 2\xi'$$

$$A = \arcsin\mathbf{A} = \text{a constant for the ship}$$

$$B = \arcsin\frac{\mathbf{B}}{1 + {}^{1}\!/_{2}\sin D}$$

$$C = \arcsin\frac{\mathbf{C}}{1 - {}^{1}\!/_{2}\sin D}$$

$$D = \arcsin\mathbf{D} = \text{a constant for the ship}$$

$$E = \arcsin\mathbf{E} = \text{a constant for the ship}$$

$$\mathbf{B} = \frac{1}{\lambda}\left(c\tan\theta + \frac{P}{H}\right)$$

$$\mathbf{C} = \frac{1}{\lambda}\left(f\tan\theta + \frac{Q}{H}\right)$$

where at a given location of the ship,

$$H = \text{the horizontal force of the magnetic field of the Earth}$$

$$\theta = \text{the dip, or inclination, of the magnetic field of the Earth}$$

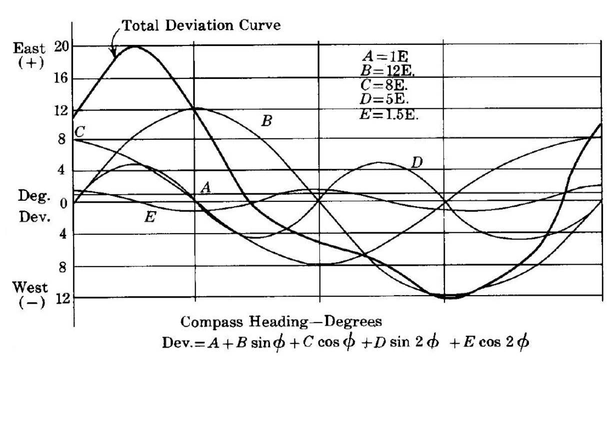
and A, D, E, λ, c, P, f and Q are parameters deduced for a particular ship.

This set of equations takes into account the magnetic effects of both hard and soft iron in the ship. Hard iron possesses permanent magnetism, while external magnetic fields such as the Earth's induce magnetism in soft iron. In essence magnetic deviation has the following components:

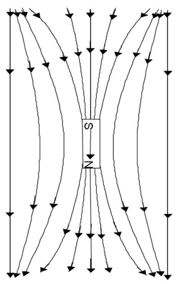
1. A constant term A due to any misalignment of compass north to the ship's keel line and to asymmetrical arrangements of soft iron horizontally in the ship.
2. **Semicircular** forces (those with a period equal to 360°):
   (a) Fore and aft components of the permanent magnetic field due to hard iron and the induced magnetism in asymmetrical vertical iron forward or aft of the compass—the $B\sin\xi'$ term.
   (b) Athwartship component of the permanent magnetic field due to hard iron and the induced magnetism in asymmetrical vertical iron port or starboard of the compass—the $C\cos\xi'$ term.
3. **Quadrantal** forces (those with a period equal to 180°):
   (a) Induced magnetism in symmetrical arrangements of horizontal soft iron—the $D\sin 2\xi'$ term.
   (b) Induced magnetism in asymmetrical arrangements of horizontal soft iron—the $E\cos 2\xi'$ term.

The terms P/H in $\mathbf{B}$ and Q/H in $\mathbf{C}$ are due to the permanent magnetism of the ship. The magnetic forces P and Q are not dependent on H, but the reciprocal of H appears because the countering force tending to keep the needle on magnetic north is given by H.

Even though the quadrantal terms $D\sin 2\xi'$ and $E\cos 2\xi'$ are dependent on the induced magnetism in



the soft iron, it turns out that they are not dependent on location. This is because the quadrantal force is proportional to H, and the force that keeps the needle on magnetic north is H, so as H changes the two forces vary in the same proportion and the net deflection is constant. The semicircular terms $B\sin\xi'$ and $C\cos\xi'$ do depend on the Earth's magnetic field, including its dip, at each ship location.



It may surprise you that the magnetic dip affects the compass—it certainly surprised me. After all, it would seem that only the horizontal component would have an effect on a horizontal compass. But in fact the dip θ is used with the horizontal component H to derive the vertical component $H\tan\theta$ which does have an effect on the induced horizontal magnetic field of the ship. The vertical component of the Earth's magnetic field bends in the presence of soft iron as shown in the figure to the left. Another way to look at this is that the net field is the superposition (the vector sum) of the vertical magnetic field and the dipole magnet induced in the iron. This creates an induced horizontal component due to the vertical field, and this will affect the compass needle. This is negated somewhat by a vertical Flinder's bar with its upper end at the level of the needle.

It's interesting to note that all this was well-known prior to Maxwell's unification of electromagnetic theory—in fact, he refers to it in his famous treatise. And the basic theory is unchanged from that time; the composite graph shown earlier of the components of magnetic deviation is from the *Handbook of Magnetic Compass Adjustment* issued by the National Geospatial-Intelligence Agency in 2004.

Other Sources of Magnetic Deviation

There is another effect not taken into account so far, the distortion due to *heeling* of the ship, i.e., the leaning of the ship from wind as well as the transient rolling of the ship. Smith derived equations for all that as well in his manual.[4]. Heeling produces a maximum deviation when the ship is heading north or south, and no deviation when heading east or west (although the needle will have less directive force to north in this case). The complementary pitching action of the ship, being more transient that heeling, does not produce a significant difference in deviation on average. Another transient effect found in practice, the *Gaussin error* (not Gaussian error), is a time lag in magnetic change with heading change of about 2 minutes due to opposing magnetic fields in the soft iron created by eddy currents (by Lenz's Law). Of greater concern is the *retentive error*, or the tendency to retain residual, subpermanent magnetism in the hard iron that is accumulated as the ship maintains a set course for a long time (say, several days) while being hammered by waves, an effect that can last from hours to more than a day after a heading change. This certainly required some experience and a good bit of art to reckon in the past.

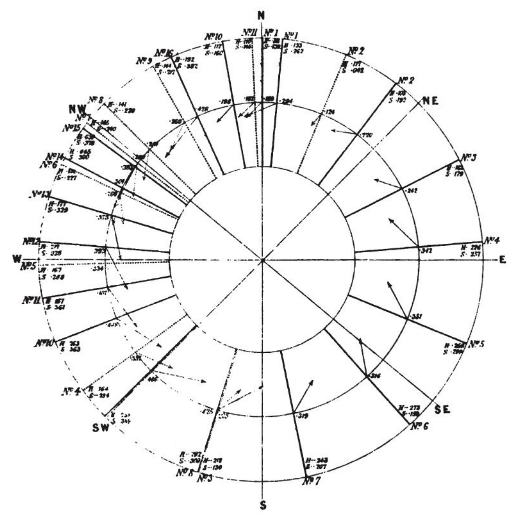
Add to this the changing effects on magnetic deviation from variable quantities of ammunition on board, varying turns on cable reels, attached boats and nearby ships, personal effects such as watches and belt buckles, stowing of the anchor chain, lightning strikes, the heating of smoke stacks and exhaust pipes, and so on. Newer aluminum boats, for example, don't provide magnetic shielding of sources below deck, and Barber relates that in one new aluminum cutter the compass deviation obediently tracked the generator speed. With all these effects it's not surprising that at one time a magnetic compass was often placed high on the mizzenmast for a sailor to climb to take readings, a very effective solution in calm seas but a problematic one when a bobbing

[4] Gray presents a nice summary of it all.

ship magnified the needle bounce and swing![5]

The direction of permanent magnetism of hard iron is related to the direction that the ship was facing when it was built; the compass needle will be attracted to the part of the ship that was south of it during construction. Smith held that an iron ship should be built with its head in a north-south direction, and preferably south. The effect is due to the alignment of magnetic domains in the iron with the external magnetic field of the Earth while being worked and pounded. In fact, Gilbert had created magnets by hammering iron rods laid in a north-south direction as part of his demonstration that the Earth acts as a mostly dipole magnet. But this initial permanent magnetism doesn't last, and in some cases over half of a ship's original permanent magnetism is lost in the course of its first year of use. And while the permanent magnetism of a ship is fairly constant after that point, any collision or repair of the ship will alter that permanent magnetism, requiring a new set of measurements and corrections to be applied.

Ascertaining Ship Parameters



The measurement of the magnetic deviation parameters for each compass location on a given ship was a tedious job that prompted many proposals for the best method. In general the dip θ at the measurement location was found on shore with a dipping needle. The horizontal component of the Earth's magnetic field H was also found at that location, normalized to 1.0 for a standard location (Greenwich, for Smith). Then the combined horizontal magnetic force of the Earth and ship (H') was found at each compass location on the ship itself. To do this, the needle of a precision compass was manually rotated to one side and released, and the time for a set number of vibrations was recorded, both on board (T') and on shore (T). Then the equation $H'/H = T^2/T'^2$ produces H'. Measurements were obtained as the ship was *swung* to different *rhumbs* of the compass rose, 8 directions for approximate values and 32 for more exact results, executed slowly and in opposite directions to reduce the Gaussin error. The difference between the compass course and the known magnetic heading, or the deviation δ, was also recorded at each rhumb. With varying levels of complexity, the required constants were extracted. For example, the *mean directive force* to magnetic north λH is found as the average value of $H' \cos\delta$.

For a sub-optimal suite of only 8 measured deviations at 45° intervals (i.e, N, NE, E, SE, S, SW, W and NW), we can find the inexact coefficients from these rules:

A is the mean of the algebraic sum of all the deviations
B is the mean of deviation at E and the negative of the deviation at W
C is the mean of the deviation at N and the negative of the deviation at S
D is the mean of the deviations at NE and SW and the negatives of the deviations at SE and NW
E is the mean of the deviations at N and S and the negatives of the deviations at E and W

[5] The construction of a compass that would minimize needle swing due to the motion of a ship was also a long-running debate, with the version by William Thomson (later Lord Kelvin) in his popular commercial binnacle eventually losing out to liquid-filled models.

For any more than 8 rhumbs the derivation of the parameters is very complicated and involves solving a complex system of equations developed by Archibald Smith and others. Smith's equation in A, B, C, D and E is actually a truncation at $2\xi'$ of a Fourier series expansion in sines and cosines of multiples of the course ξ', so these values are the corresponding Fourier coefficients—in fact, the first use of the phrase *harmonic analysis* is found in William Thomson's obituary of Archibald Smith [Grattan-Guinness]. For more detailed information on deriving the ship's parameters in this way, or for deriving the exact coefficients **A**, **B**, **C**, **D** and **E**, please see Smith's *Admiralty Manual for the Deviations of the Compass* or *The Magnetism of Ships and the Deviations of the Compass, Comprising the Three Reports of the Liverpool Compass Commission*.

DEVIATION TABLE.

(For ship's use.)

Form No. 12 (a).

......................* *Compass* ....

U. S. S... ....

Date.., *191*

Latitude.............. *Variation used*....................

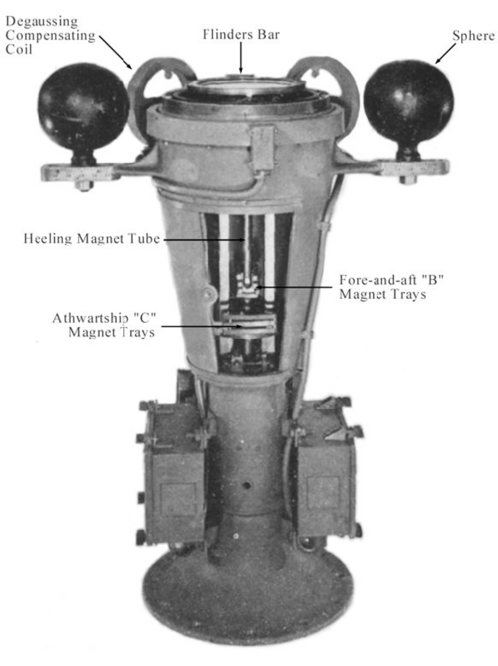
Longitude..

Ship's head by compass	Deviation	Ship's head by compass	Deviation
	° ′		° ′
0°	0 0	180°	0 00
15°	+ 2 00	195°	+ 9 00
30°	+ 0 30	210°	+ 16 30
45°	− 3 00	225°	+ 21 00
60°	− 8 00	240°	+ 22 45
75°	− 14 00	255°	+ 21 15
90°	− 19 30	270°	+ 17 30
105°	− 23 00	285°	+ 12 00
120°	− 25 00	300°	+ 7 00
135°	− 24 30	315°	+ 2 00
150°	− 21 00	330°	− 1 30
165°	− 13 00	345°	− 2 00

APPROVED: .... *U. S. Navy, Commanding.*

Compensating for the Magnetic Deviation

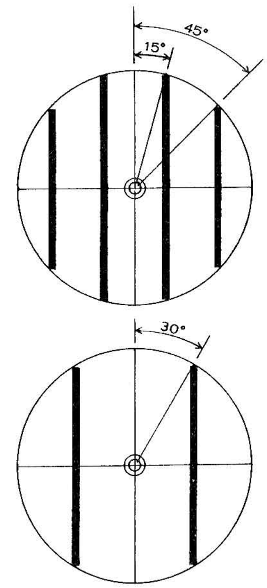
The magnetic deviation of a ship is typically corrected, even today, by components located in the binnacle holding the compass. Permanent magnets are positioned to compensate for the permanent magnetism of the ship. A vertical soft iron bar (the Flinders bar) is also located near the compass to counter the effects of the vertical component of the Earth's magnetic field. These correct for the semicircular forces. Soft iron spheres on a rotating base serve to correct for the quadrantal forces, but their positions have to adjusted for the magnetic latitude.



The spheres also help negate magnetic deviations from heeling of the ship. There are also adjustable permanent magnets included to overcome these heeling effects. A permanent magnet mounted vertically directly beneath the compass does not have any effect when the ship is upright, but will correct for heeling error as the compass needle dips a bit in the lean. These permanent magnets also need to be adjusted with latitude.

Finally, there are current-carrying coils in the binnacle that are energized to counter the effects when the ship activates its degaussing coils to elude mines that trigger on the magnetic fields of passing ships.

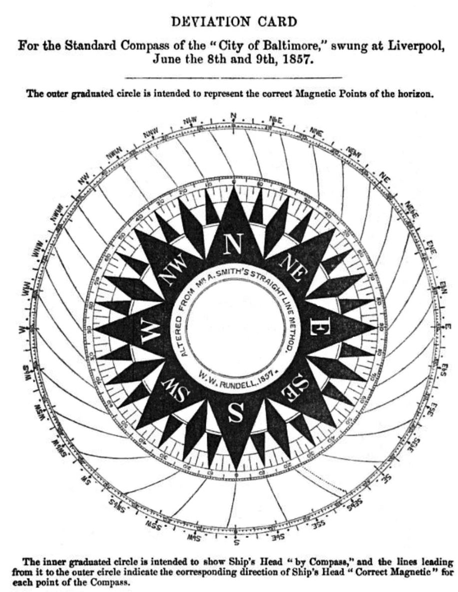
Occasionally the net effect of magnetic deviation on an uncompensated compass completely negates the magnetic effect from the Earth, and the compass has no preferential direction at all, or only a weak one that makes observations uncertain. For this reason compensation is usually preferable to simply adding a correction in degrees from a table or diagram. I might add that Thomson once said that the chances were 50-50 that the navigator would get the sign wrong in calculating a compass correction [Barber]. Also, the equations given earlier assume a magnetic deviation of less than about 20° in order that B and C can be expressed as simple arcsine functions, and so a certain amount of ship correction is generally needed in an iron ship to ensure this.



For centuries, long compass needles (say, up to 15 inches) were thought better for higher magnetic strength (true enough) and better stability in rough seas (not true at all). But it happens that there are sextantal deviation terms in $3\xi'$ for these long compass needles due to their response to the permanent magnet compensators, and octantal terms in $4\xi'$ due to the interaction of these needles with their magnetic images in soft iron compensators. Smith early on had developed a rule for compass card needles, that when there are two needles they should be placed with their ends on the card at 30° to each side of the ends of the north-south diameter of the compass (the lubber line), and when there are four needles they should be placed with their ends at 15° and 45° from the ends of the diameter (30° apart). In this way there are equal moments of inertia parallel and perpendicular to the compass axis which eliminates wobbling of the card. But twenty years later, in analyzing ship data exhibiting the higher-order effects, Evans and Smith discovered that small needles arranged in just the way he had prescribed exhibited less sextantal and octantal deviation than one short needle—and exactly zero mathematically! It was "a happy coincidence" according to Smith, and this justified moving the compensating permanent magnets and soft iron correctors much nearer this type of compass for more accurate elimination of the semicircular and quadrantal deviations.[6]

Computing the Magnetic Deviation

William Thomson called them "beautiful and ingenious geometrical constructions," and in variance to their rather humdrum name dygograms are certainly charming to the eye. But these geometric constructions can conveniently generate and then calculate the magnetic deviation of a ship compass at a location.

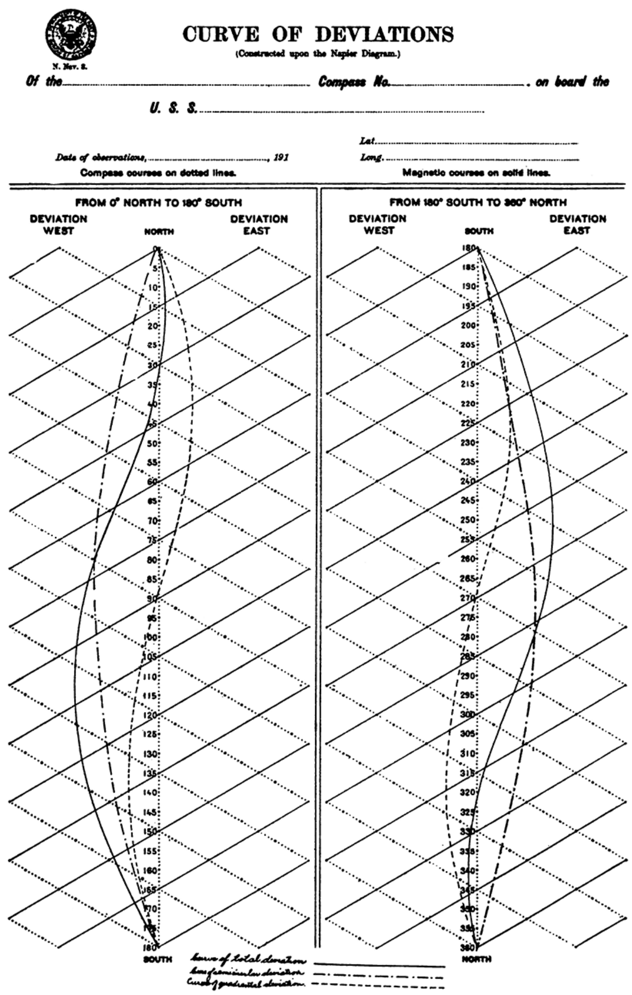


With our electronic calculators and computers, we take for granted the effortless arithmetic and trigonometric calculations that so vexed our ancestors. Pre-calculated tables for roots and circular functions, generated through hard work, were often used to create tables of magnetic deviations for specific ships and locations. To reduce the chance of misreading these tables, a few types of graphical diagrams, not just dygograms, were invented to provide fast and accurate readings of magnetic deviation. These graphical calculators are the focus of this part of the essay.

One such chart, or deviation card, is shown in the figure on the right. Here the sailor would follow his compass heading on the inner circle and arrive at a course on the outer circle that is corrected for the magnetic deviation of the ship at the location for which the card was constructed.

[6] See Lyons for detailed proofs of these.

A more typical chart invented in 1851 is called a *Napier's diagram* after James Napier (1821-1879). The simple nature of this chart belies its advantages in obtaining a fast and reliable correction. To create it, vertical linear scales of compass courses are drawn (two for good resolution), along with with dotted and solid lines at 60° angles from the scales as shown in the figure (originally these were at 90° and 45°). Then the magnetic deviation is plotted opposite the scale, left (for west) or right (for east) such that the deviation for a given course is located on the solid line running 60° from the scale value. The semicircular and quadrantal components are also individually plotted as dashed curves in the ones I have seen. To read the correction, you would find the compass heading on the scale, proceed up the dotted line to the solid curve, then back to the scale along the solid line (or parallel to them for intermediate scale values). Then since the triangles are all equilateral, you've directly found the corrected compass heading without doing any arithmetic. Simple in design, robustly effective in use!



Napier's diagrams could be directly plotted from data obtained as a ship was swung, and much was made of the then-new method of least squares in drawing the curves. It can be seen here that the solid curve is the sum of the plotted semicircular curve, the quadrantal curve, and the constant deviation *A* when significant. Only the semicircular deviation and the new total need to be re-plotted for other locations.

As ship design evolved from simple iron plating to iron hulls with larger engines, the inaccuracy of the inexact coefficients *A*, *B*, *C*, *D* and *E* became noticeable. On the other hand, the equation expressing the magnetic deviation in terms of the exact coefficients **A**, **B**, **C**, **D** and **E** was difficult to compute, despite a set of mathematical tables and rules specifically prepared for this purpose by Smith. In an attempt to deal with this, Smith invented the geometric constructions he called **dygograms** to provide a graphical calculation of the magnetic deviation for any compass course using the exact rather than inexact coefficients.

Dygograms

The *Admiralty Manual* describes in detail two main types of dygograms, called Dygogram I and Dygogram II. We will briefly investigate each of these.

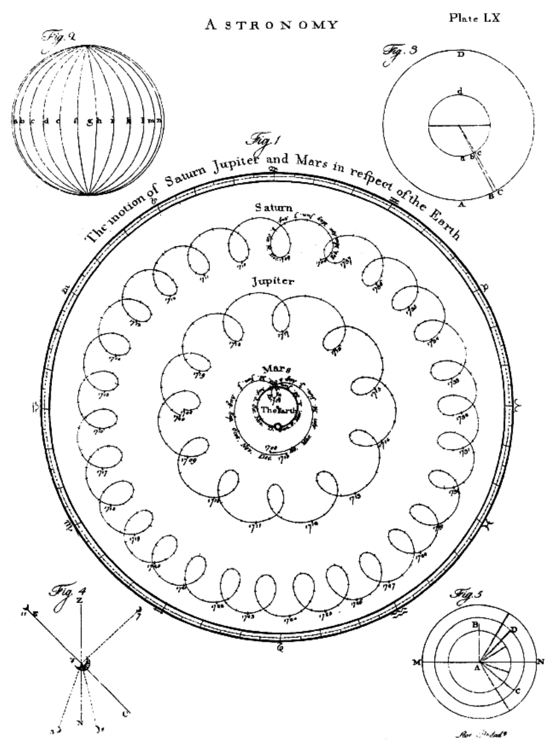
To create a dygogram for a given ship, it is assumed that the exact coefficients **A**, **B**, **C**, **D** and **E** are known. However, Smith does derive inverse series for extracting these exact coefficients from the inexact coefficients *A*, *B*, *C*, *D* and *E* that can be obtained by harmonic analysis from measurements taken as the ship is swung.

Dygogram I

Referring again to Equation (2) expressed in terms of the inexact coefficients,

$$\delta = A + B\sin\xi' + C\cos\xi' + D\sin 2\xi' + E\cos 2\xi'$$

we see that the magnetic deviation consists of a constant value, semicircular terms with a period of 360°, and quadrantal terms with a period of 180°, each constant in magnitude. So as the ship swings completely around, the semicircular contribution make one revolution while the quadrantal contribution makes two revolutions. For the Dygogram I type, Smith modeled the net deviation as one point revolving on a circle at a certain angular speed, with a second point revolving on an outer circle centered on the first point but at *half* that speed. With correct scaling, the inner circle represents the contribution of the quandrantal terms while the outer circle represents the contribution of the semicircular components. An offset of A between the inner circle's center and the origin completes the model.



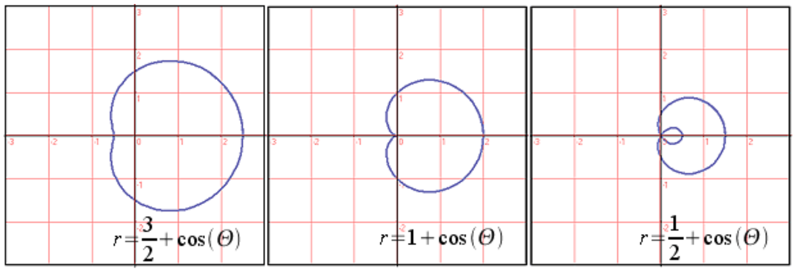
This is an epicyclic motion similar to the Ptolemaic epicycles that modeled positions of the outer planets, except that Ptolomy had the outer circles rotating faster than the inner ones. This can also be modeled as a point on a circle as the circle rolls around the circumference of another circle, where here the effective radius of the inner circle is increased by the radius of the outer circle. This is akin to a penny rolling around another penny, in which Abraham Lincoln rotates twice for every revolution of the outer coin. Smith originally assigned the faster quadrantal rotation to the outer circle, and to me this is the instinctive way to do it, but by the 3^{rd} edition of the *Admiralty Manual* he reversed these to take advantage of simplifications in construction proposed by Lieut. Colongue of the Russian Imperial Navy. It's also a fact that the quadrantal force does not depend on the ship's location, so he could affix this as the inner circle and then redraw only the outer curve for various locales.

The overall curve that results from a point rotating on a circle that is revolving around another point rotating on a circle is called *The Limaçon of Pascal* after its discoverer Étienne Pascal, father of Blase Pascal. It was named (from the Latin *limax* for snail) by Gilles-Personne Roberval in 1650 in his use of it to draw tangents as a means of differentiation. It has the general parametric form

$$x = b\cos\theta + a\cos 2\theta$$
$$y = b\sin\theta + a\sin 2\theta$$

or in polar coordinates,

$$r = b + a\cos\theta$$



For a particular ship and location, the limaçon can be a circle if $a = 0$, it can approach but not touch the inner circle (as a dimpled circle) if $a < b < 2a$, it can just touch the inner circle (as a cusped curve, or a cardioid) if $a = b$, or it can loop within the inner circle and exit it if $a > b$. It's a popular curve. As Smith describes it for $p = b$ and $q = a/2$,

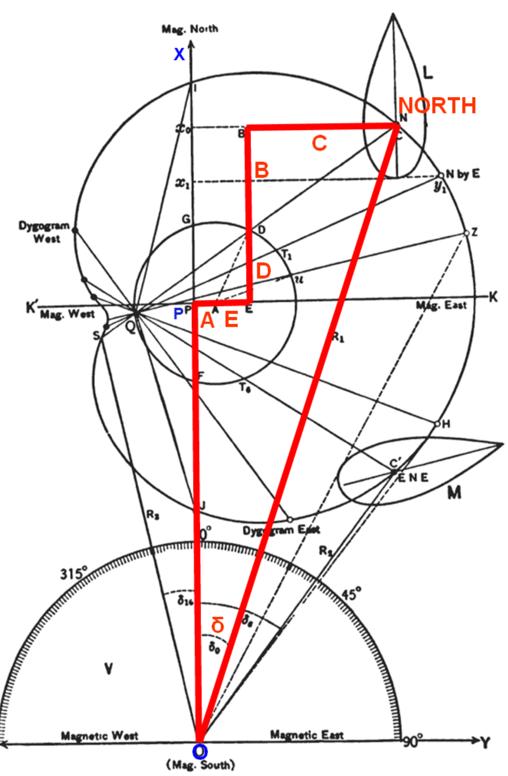
> It is at once a conchoid of the circle, an epitrochoid, and a Cartesian oval. When $p = q$ it gives, as was shown by Pascal, a solution of the problem of the trisection of the circle; when $p = 2q$ it is a caustic by reflexion of the circle.

A Type I dygogram is shown on the following page [from Lyons]. We will be using miniature versions of it as we go. The dygogram calculates the magnetic deviation δ from the following equation in terms of the exact coefficients and the magnetic course ξ:

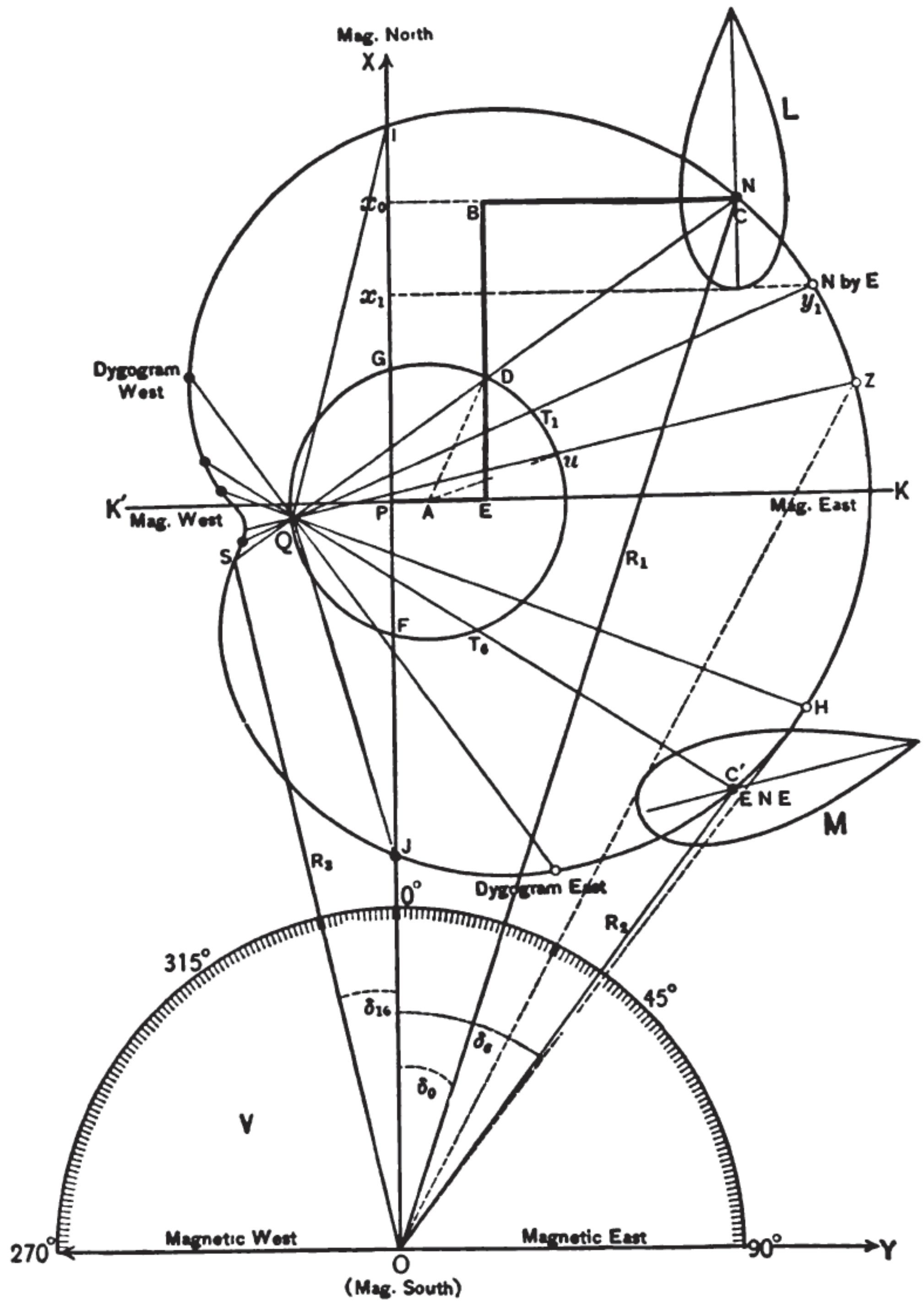
$$\tan\delta = \frac{\mathbf{A} + \mathbf{B}\sin\xi + \mathbf{C}\cos\xi + \mathbf{D}\sin 2\xi + \mathbf{E}\cos 2\xi}{1 + \mathbf{B}\cos\xi - \mathbf{C}\sin\xi + \mathbf{D}\cos 2\xi - \mathbf{E}\sin 2\xi} \qquad (3)$$

Now when the magnetic course is due north, we have $\xi = 0°$ and the formula reduces to:

$$\tan\delta = \frac{\mathbf{A} + \mathbf{C} + \mathbf{E}}{1 + \mathbf{B} + \mathbf{E}} \qquad (4)$$

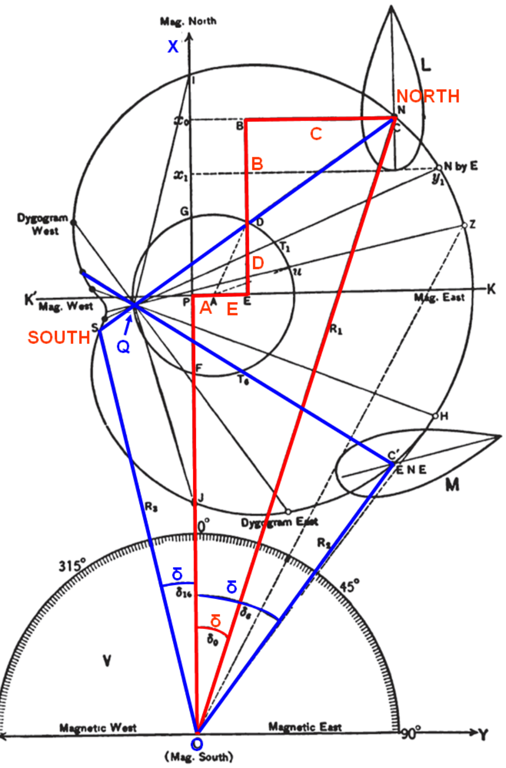


Referring to the dygogram components in red in the figure to the right, we place a point O at the bottom of the page and at a convenient distance above it we place a point P. This distance is defined as the length of 1 in the dygogram and represents the mean directive force to north λH. Then using this as a unit length we move right a distance $\mathbf{A}$ to plot the point A, right again a distance $\mathbf{E}$ to plot the point E, up a distance $\mathbf{D}$ to plot the point D, up again a distance $\mathbf{B}$ to plot the point B, and right a distance $\mathbf{C}$ to plot the point C. From the figure you can see that Equation (4) for $\tan\delta$ with $\xi = 0°$ holds for this diagram, and therefore if magnetic north lies along the vertical line from O, that the angle from it to C (which is marked NORTH here) is the magnetic deviation when the ship is pointing north. So we mark this point C as N for the north magnetic course $\xi = 0°$ and for clarity draw a ship pointing north at this point.



Example presented in this essay of the Dygogram I type [from Lyons].

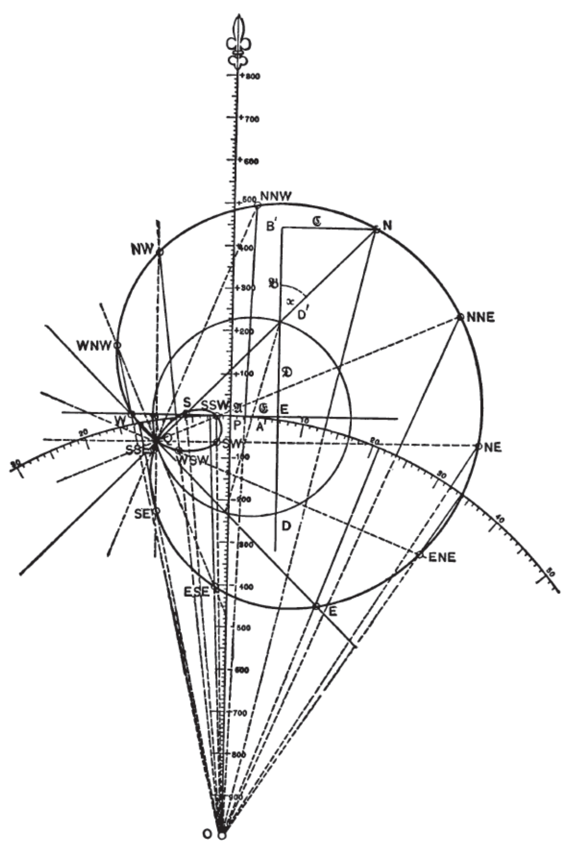
Now consider the blue markings we've added to our dygogram figure. For a magnetic course of due south, $\xi = 180°$, and from Equation (3) the only effect is that **B** and **C**, the distances that define the outer curve, change sign. So we extend (the math term is *produce*) ND a distance equal to ND in an opposite direction from point D and we get the appropriate point S on the curve that will provide at O the magnetic deviation for a magnetic course of due south, that is, the angle between the vertical line OX and the line OS.



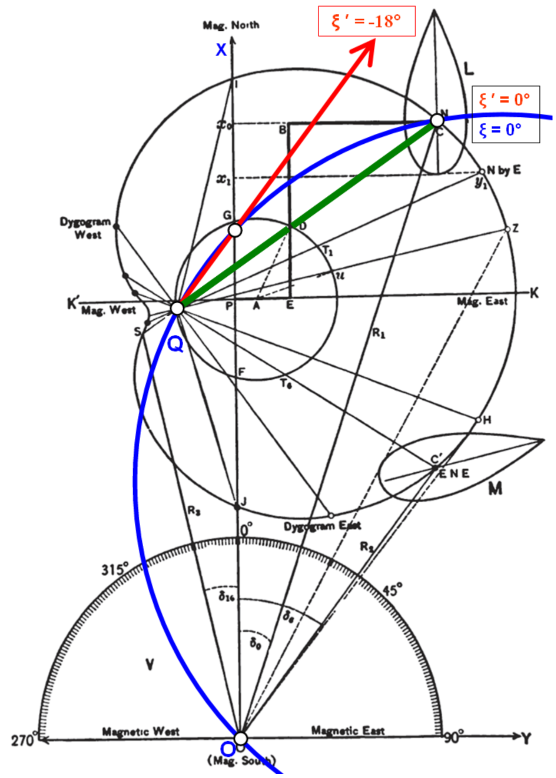
Now from the order in which we have placed these coefficients horizontally and vertically on the diagram you can see that the radius of the inner circle AD equals the magnitude of the sum of the quadrantal terms, or $\sqrt{\mathbf{D}^2+\mathbf{E}^2}$. The distance DC equals the magnitude of the sum of the quadrantal terms, or $\sqrt{\mathbf{B}^2+\mathbf{C}^2}$. The distance **A** simply produces a constant offset angle.

So let the magnetic course ξ proceed clockwise around from the north point N. If we rotate the constant radius AD by 2ξ about the point A while we rotate the constant distance DC by ξ around D, then the horizontal distance from the vertical line OX of the point on the curve will equal $\mathbf{A}+\mathbf{B}\sin\xi+\mathbf{C}\cos\xi+\mathbf{D}\sin 2\xi+\mathbf{E}\cos 2\xi$ while the vertical distance above O will equal $1+\mathbf{B}\cos\xi+\mathbf{C}\sin\xi+\mathbf{D}\cos 2\xi+\mathbf{E}\sin 2\xi$. Therefore, the tangent of the angle at O will be the ratio of these, and by Equation (3) the angle at O between the vertical line and the line to the point on the curve is the magnetic deviation δ for that magnetic course.

A limaçon is the curve that results from these required relationships. We mark a point Q where the north-south line NS intersects the inner circle. As the angle ξ rotates about Q, the segment DC will rotate at the same rate but the angle DAE will rotate at twice the angle. The point Q is called the pole of the dygogram and ξ is measured clockwise about it from QN. You can see on the figure that east-northeast is marked at a position 67.5° clockwise from N.

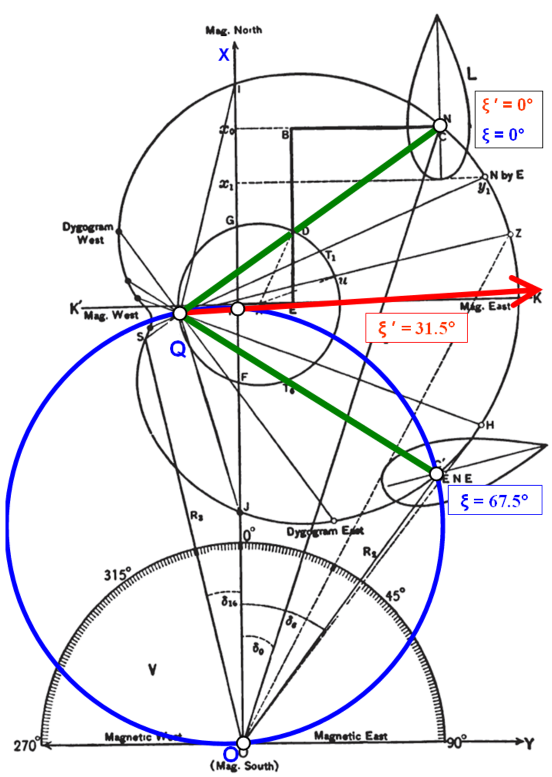


We construct the outer curve by laying a straightedge along NS and marking the positions of N, D and S on it, where D will be the midpoint of NS. We always need DC=DN to be a constant length from D, so we place the straightedge at various locations with D lying on the inner circle and passing through Q. At each location we mark points on the paper corresponding to N and S on the straightedge and we connect them to construct the curve. Then with a protractor centered at Q we mark various cardinal and intercardinal points of the compass on the outer curve and the dygogram is complete. The outer curve can indeed loop within the inner circle in some cases, as seen in the other dygogram here. In practice most of the construction lines are removed in the final dygogram. From this graphical calculator we can now easily find the magnetic deviation for any magnetic course of the ship.



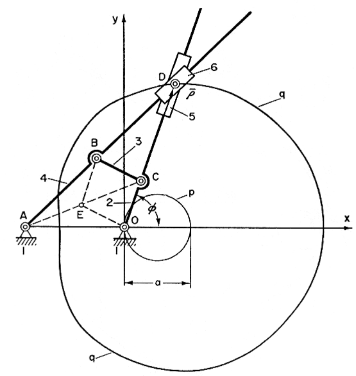
But again we would like to have the deviation in terms of the compass course ξ' that we are reading on the ship. We know that $\delta = \xi - \xi'$, so we can calculate from the dygogram the deviation for a given magnetic course and find the compass course as $\xi - \delta$, so we have the compass course for that magnetic course, but finding the deviation from a known compass course would be a trial and error process. Smith recommends laying out a Napier's diagram for all the deviations, plotted offset along the solid lines from the compass course scale, and then using that diagram for any desired compass course. Smith also provides a couple of constructions to approximate the magnetic course for a desired compass course. But the best construction he describes is again due to Lieut. Colongue.

To demonstrate how we can find the magnetic deviation from the compass course, let's first work backwards from the sample dygogram we've been using. In the figure on the right, for a north magnetic course of $\xi = 0°$ we can read a magnetic deviation $\delta = 18°$ on the protractor (the angle between the vertical line OX and the north position of the ship in the upper right). Since $\delta = \xi - \xi'$ we find that $\xi' = -18°$. Now what we are going to do is to start with a compass course $\xi' = -18°$ and see if we end up calculating the same deviation $\delta = 18°$.



The method is to first draw the (red) line from Q at the compass course angle ξ from the (green) line QN and mark the point where it intersects the vertical line OX. Then with dividers we draw the arc of a circle that passes through three points: O, Q and this intersection point. It will intersect the dygogram curve at the magnetic course ξ, and we proceed as before to find the deviation by the angle between OX and a line from O to that point. As you can see, it works out perfectly here, as that intersection corresponds to the north magnetic course that gave us $\delta = 18°$. Using dividers to find the arc is analogous to trial and error, I suppose, but it's a lot easier than doing math by trial and error, and this is one big advantage of graphical calculators in general.

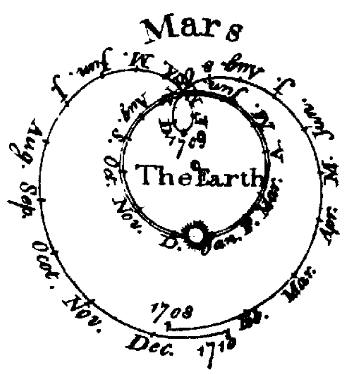
Again, let's verify it for the ENE location marked on the dygogram to the left. The magnetic course for this intercardinal point is 67.5° around Q from the north point N. We read from the protractor at the bottom that the deviation for this magnetic course is 36°, so we start with $\xi' = 67.5° - 36° = 31.5°$. We draw a (red) line from Q at this angle from the (green) line QN and mark its intersection with OX. We draw the arc connecting O, Q and this point, and it indeed intersects the dygogram curve at the ENE point where $\xi = 67.5°$, from which we can read the deviation. So these two examples demonstrate that we can find the deviation for any compass course on the dygogram with a few extra steps.



Once the dygogram is constructed for one location, Smith provides simple procedures for plotting the dygogram at any other location from as few as two measurements of deviation vs. magnetic course at that place. In the process the values of the location-specific coefficients **B** and **C** are also found. Smith suggests a mechanical way of tracing the curve with a roller revolving about a roller in the same way as a coin revolves around the coin, but there are other ways to custom-draw a limaçon; I came across this mechanism in a book on linkages.

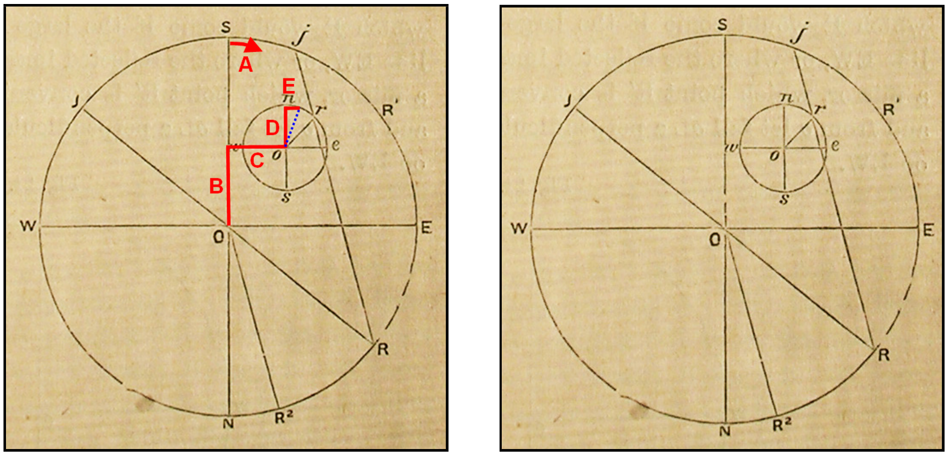
Dygogram II

Smith created an intermediate type of dygogram that consisted of an ellipse and a circle in lieu of the limaçon, but we are going to proceed to his ultimate form consisting of just two circles. I have only seen this discussed in the *Admiralty Manual* and in Smith's obituary by Thomson, and only Thomson describes the case where **A** and **E** are non-zero. It is a clever transformation.



Smith relates that it occurred to him to turn the paper with the same velocity that the ship turns, i.e., at the half-speed rotation of the outer generating circle, while tracing the limaçon in a same direction. Then the fixed point Q traces an inner circle, and the limaçon also ends up tracing another, outer circle. Here's my guess as to how he might have come to that. Let's zoom in on the innermost orbits in the figure of the Ptolemaic system we saw earlier. Here we see, from an Earth-centered point of view, the Sun revolving in a circular orbit around the Earth and Mars circling in what appears to be a perfect dygogram! And it very nearly is—it certainly is an epicycloid. If Mars were to have an orbital period of 2 Earth-years instead of 1.88 Earth-years it would be a dygogram, because the Sun point is modeled as revolving about the Earth with a period of 1 year and to match observations Mars must be modeled as revolving about the Sun point with a period of 2 years. Now as we know, Copernicus demonstrated that by changing the reference frame to a Sun-centered system (by fixing the paper on the Sun as the orbits trace) we find that the orbits of Earth and Mars end up as simple nested circles. We have an analogous situation here.

To construct this type of dygogram, a circle of radius 1 of some unit of length is drawn with a center O (see the next page). Then using this unit length we move up from O a distance **B** and right a distance **C** and mark this point *o*. Then we move up a distance **D** and right a distance **E**. We draw a circle with its center at *o* that passes through this last point. This small circle is marked *n*, *e*, *s* and *w* and degrees are marked on it in a clockwise direction. The large circle is marked S, E, N and W but the position of these is rotated by **A**. Degrees are marked on it counterclockwise from north. Again Smith provides methods of plotting such a dygogram for different locations from a few observations.



Dygogram II construction (left) and final configuration (right).

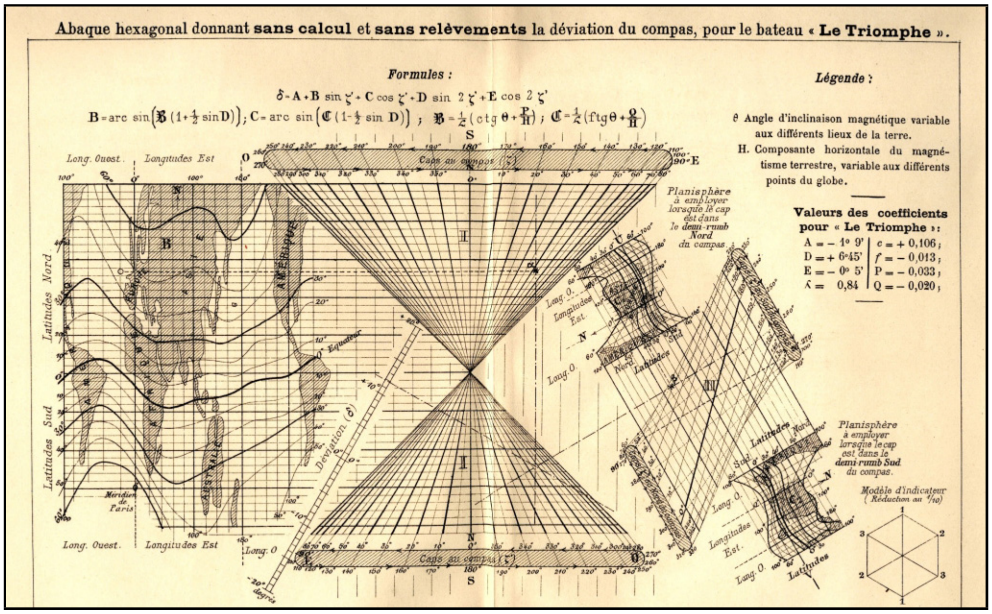
Once the dygogram is created, Smith provides the following succinct procedure for reading the magnetic deviation for a given magnetic course:

> Let ξ be the given magnetic course. Take R, a point on the circumference of the large circle, and *r* on the circumference of the small circle, such that NOR = *nor* = ξ, and join OR, R*r*. Then OR*r* is the deviation which is $+$, or easterly, if *r* is to the right of O looking from R to O. If the large circle is graduated we may measure the angle OR*r* by producing RO, R*r*, to intersect the circle in J and *j*. The arc J*j* will then be twice the required angle.

But, again, on the ship we only know the compass course ξ' we are reading from our compass. To use this we can place a straightedge that intersects the center O of the large circle and the value of ξ' on the outer circle. Then we move it parallel to itself (there are linkages for parallel rulers) until it intersects the two circles at the same marked angle. These are the points R and *r* with values equal to the magnetic course ξ for this compass course, and we proceed from here as before to find the magnetic deviation. Again we see the power of a graphical calculator to naturally close in on a solution that would otherwise be a tedious trial-and-error arithmetic calculation.

Smith describes drawing a large set of lines between points on the two circles that have corresponding marked angles to provide a convenient, overall visual layout. Also, if the large circle is considered to be an upside down compass card, we can glue the (upside-down) small circle onto the compass card itself. In those days the needle was attached to the compass card, so the card turns with the needle and the compass course is the reading of the card in the forward direction the ship is facing. When the ship is at sea, we can find which drawn line between the two circles is fore-and-aft (which will be parallel to the compass course shown on the fore edge of the rotated card), and this will cut the two circles in points corresponding to the magnetic course. Or better yet, we can steer a magnetic course by turning the ship until the line connecting the desired magnetic course values on the two circles is fore-and-aft. Smith termed this a **steering dygogram**. It seems absolutely brilliant to me, but the lack of ready literature on it suggests it was never really taken up.

These graphical calculators—the Napier's diagram and the various incarnations of the dygogram—are convenient devices for obtaining the magnetic deviation of a particular ship at a particular place. But a ship at sea would have to carry a set of charts like these for various locales, one more variable that could lead to disastrous errors. However, in 1885 a French engineer named Charles Lallemand created a uniquely designed and somewhat famous graphic, a hexagonal chart of his own invention, to calculate the magnetic deviation of the ship *Le Triomphe* no matter where it was located. The design and workings of this chart is the subject of another essay of mine, *Lallemand's L'Abaque Triomphe, Hexagonal Charts, and Triangular Coordinate Systems*.



Lallemand's 1885 hexagonal chart for computing the magnetic deviation of the ship *Le Triomphe* at any location.

A Brief "Dygression"

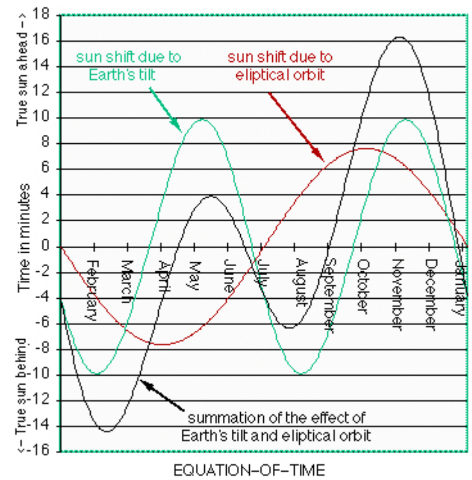
In an article in 1907 A.G. Greenhill used a dygogram to model the reaction force on the axle of a pendulum. And just now as I write the final paragraphs of this essay, I look at the steering dygogram and it reminds me of sun compasses used as very early navigational aids. And that reminds me of portable sundials, which sparked my interest in nomography and graphical calculators in the first place. And *that* reminds me that the Equation of Time correction (the difference between the mean clock time and the actual solar time that must be accounted for in any good sundial) is composed of a sinusoidal term with a period of a year and a sinusoidal term with half that period!

Why this didn't occur to me before is a mystery. During the few months I've been researching this essay, including reading through all the references and buying the only copy of Smith's *Admiralty Manual* I could find in this hemisphere, I've also been plotting figure-8 analemma curves to represent the Equation of Time (EOT) correction on a sundial I'm designing for my house.

The non-circularity (or eccentricity) of the Earth's orbit is one component in the EOT. The other component is the angle between the equator and the plane of the orbit (or the ecliptic tilt). Whitman provides a relatively simple but accurate formula for the EOT for a given day:

$$\Delta t = 229.18\left[-0.0334\sin\left(\frac{2\pi}{365.24}(t-t_0)\right)+0.04184\sin\left(\frac{4\pi}{365.24}(t-t_0)+3.5884\right)\right]$$

where Δt is the correction in minutes and $t-t_0$ is the number of mean (clock time) solar days since the Earth's perihelion (or closest approach to the sun).



The perihelion date varies with the year depending mostly on leap year differences, and it ranges from Jan. 3 to Jan. 5, so we can let $t_0 = 4$. The EOT is typically averaged over the four-year leap day cycle anyway. Then let $\alpha = {}^{2\pi}/_{365.24}$ and we have

$$\Delta t = -7.655\sin(\alpha - 0.0688) + 9.589\sin(2\alpha + 3.451)$$

We can twice apply the identity

$$\sin(a \pm b) = \sin a\cos b \pm \cos a\sin b$$

and find that

$$\Delta t = -7.637\sin\alpha + 0.526\cos\alpha - 9.134\sin 2\alpha - 2.920\cos 2\alpha$$

which is in the form of the magnetic deviation equation for the inexact coefficients $A = 0$, $B = -7.637$, $C = 0.526$, $D = -9.134$ and $E = -2.920$.

So a dygogram can model the EOT if the 365.24 days of the year are spaced out equally through the 360 degrees (about Q in Dygogram I and along the outer circle in Dygogram II). Now in an equatorial sundial the hour lines are also equally spaced around the circumference of a circle. So it might be possible to adapt the steering dygogram to provide the EOT correction right on the sundial. I will have to think about that.

An Exquisite Endeavor

The centuries it took to untangle the mysteries of magnetic deviation represent an enormous, sustained effort by scientists, mathematicians, ship captains and crews. Many people provided the mathematical and scientific tools and data needed to analyze the problem, and in fact this essay has focused only on contributors in the West. Captain Flinders, Edmund Halley, George Airy, William Scoresby, Archibald Smith and F. J. Evans sparked major advances in this area before gyrocompasses, ring gyros, and now LORAN and GPS diminished the importance of the compass. But it was so important then—a friend of his remarked that Smith was "penetrated by the conviction of the usefulness of his work." Through his *ténacité passionnée* Smith produced a mathematical framework that defined magnetic deviation in a new and practical way, an achievement so beneficial that by the time he died, as Thomson relates,

> From every ship in Her Majesty's Navy, in whatever part of the world, a table of observed deviations of the compass, at least once a year is sent to the Admiralty, and is therefore subjected to [Smith's] harmonic analysis.

From my point of view, the heroic efforts made by these men to overcome the deadly consequences of magnetic deviation comprise a very heartening thread of history and an inspiring illustration of the role of the mathematical sciences in advancing our civilization.

Appendix: Smith's Derivation of the Basic Equations of Magnetic Derivation

The *Admiralty Manual for the Deviations of the Compass* presents in great detail Archibald Smith's derivations of the various equations for magnetic deviation. Here we will briefly review the derivation for the simplest case, that of a ship on an even keel. The derivation is given in a number of the cited references of this essay, and the presentation here is a concise merger of these.

Let the magnetic force of the Earth relative to the ship be X, Y and Z in the forward, starboard and downward directions, and the total magnetic force of the ship and Earth be X', Y' and Z'. In 1824 Siméon Denis-Poisson gave the following general equations relating these quantities, in which the induced magnetic forces from soft iron are proportional to the Earth's magnetic forces:

$$X' = X + aX + bY + cZ + P \tag{5}$$
$$Y' = Y + dX + eY + fZ + Q \tag{6}$$
$$Z' = Z + gX + hY + kZ + R \tag{7}$$

Archibald Smith added the constant terms (P, Q and R) to account for the ship's permanent magnetism due to hard iron, as Poisson considered these effects to be scattered without a significant overall impact, which was true for wooden ships of the time. Poisson also assumed certain symmetries in the iron of the ship that Smith did not.

Now assume that the ship's head is pointing an angle ξ from magnetic north at its location (the *magnetic course*), and an angle ξ' as shown on the compass (the *compass course*). The magnetic deviation δ of the compass needle from magnetic north due to the ship is then $\xi - \xi'$.

For this derivation,

H = the horizontal force of the magnetic field of the Earth $= \sqrt{X^2 + Y^2}$

H' = the combined horizontal force of the magnetic field of the Earth and ship $= \sqrt{X'^2 + Y'^2}$

θ = the dip, or inclination, of the magnetic field of the Earth

The vertical force of the Earth's magnetic field Z does not appear explicity in this derivation as it can be represented by $H \tan\theta$. It is apparent that

$$X = H\cos\xi$$
$$Y = -H\sin\xi$$
$$Z = H\tan\theta$$
$$X' = H'\cos\xi$$
$$Y' = -H'\sin\xi$$

and substituting these into Equations (5) and (6) we have

$$H'\cos\xi = (1+a)H\cos\xi - bH\sin\xi + cH\tan\theta + P \tag{8}$$
$$-H'\sin\xi = (-\sin\xi + d\cos\xi)H - eH\sin\xi + fH\tan\theta + Q \tag{9}$$

Let's multipy the first of these by $\sin\xi$ and the second by $\cos\xi$ and add them. After some simplification we arrive at

$$\frac{H'}{H}\sin\delta = \frac{d-b}{2} + \left(c\tan\theta + \frac{P}{H}\right)\sin\xi + \left(f\tan\theta + \frac{Q}{H}\right)\cos\xi + \frac{a-e}{2}\sin 2\xi + \frac{b+d}{2}\cos 2\xi \qquad (10)$$

This provides the directive force to magnetic east due to the ship in units of the Earth's horizontal force, and it has a mean value of $(d-b)/2$. The following identities were used to obtain this form of the equation:

$$\cos^2\xi = \frac{1+\cos 2\xi}{2}$$
$$\sin^2\xi = \frac{1-\cos 2\xi}{2}$$
$$\sin\xi\cos\xi = \frac{\sin 2\xi}{2}$$

Now we multiply Equation (8) by $\cos\xi$ and Equation (9) by $\sin\xi$ and subtract them. After a similar simplification we arrive at

$$\frac{H'}{H}\cos\delta = 1 + \frac{a+e}{2} + \left(c\tan\theta + \frac{P}{H}\right)\cos\xi - \left(f\tan\theta + \frac{Q}{H}\right)\sin\xi + \frac{a-e}{2}\cos 2\xi - \frac{b+d}{2}\sin 2\xi \qquad (11)$$

This provides the directive force to magnetic north due to the Earth and ship in units of the Earth's horizontal force, and it has a mean value of $1+(a+e)/2$, which we will denote as λ, so λH is the mean force to magnetic north.

Let's define the following constants:

$$\mathbf{A} = \frac{d-b}{2\lambda}$$
$$\mathbf{B} = \frac{1}{\lambda}\left(c\tan\theta + \frac{P}{H}\right)$$
$$\mathbf{C} = \frac{1}{\lambda}\left(f\tan\theta + \frac{Q}{H}\right)$$
$$\mathbf{D} = \frac{a-e}{2\lambda}$$
$$\mathbf{E} = \frac{b+d}{2\lambda}$$

These constants are in bold font here, but historically they are almost always printed in German (Blackletter) font as the hard-to-distinguish $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ and $\mathfrak{E}$. If we divide Equations (10) by (11) we get

$$\tan\delta = \frac{\mathbf{A} + \mathbf{B}\sin\xi + \mathbf{C}\cos\xi + \mathbf{D}\sin 2\xi + \mathbf{E}\cos 2\xi}{1 + \mathbf{B}\cos\xi - \mathbf{C}\sin\xi + \mathbf{D}\cos 2\xi - \mathbf{E}\sin 2\xi} \qquad (12)$$

This is the *exact equation* for the deviation δ in terms of the magnetic course ξ, constants **A**, **D** and **E** that are dependent only on the arrangement of soft and hard iron in the particular ship, and constants **B** and **C** that are dependent on the Earth's magnetic field at the ship's location as well as the arrangement of its soft and hard iron.

But at sea we really don't know the magnetic course ξ of the ship's head, just the compass course ξ' that we are reading. This is a real problem, because even when we substitute $\xi - \xi'$ for δ we don't have a closed form solution for finding ξ (and therefore δ). It's much better to find a way to express the deviation δ directly in terms of the compass course ξ'. In the process of doing this, we will end up replacing the exact coefficients **A**, **B**, **C**, **D** and **E** with inexact coefficients A, B, C, D and E.

We can do this in the following manner. First, we substitute $\xi' + \delta$ for ξ and ${}^{\sin\delta}/_{\cos\delta}$ for $\tan\delta$ in Equation (12). After some simplification we get

$$\sin\delta = \mathbf{A}\cos\delta + \mathbf{B}\sin\xi' + \mathbf{C}\cos\xi' + \mathbf{D}\sin(2\xi' + \delta) + \mathbf{E}\cos(2\xi' + \delta)$$

which can also be expressed as

$$\sin\delta = \frac{\mathbf{A}\cos\delta + \mathbf{B}\sin\xi' + \mathbf{C}\cos\xi' + \mathbf{D}\sin 2\xi' + \mathbf{E}\cos 2\xi'}{1 - \mathbf{D}\cos 2\xi' + \mathbf{E}\sin 2\xi'}$$

It turns out that **E** is relatively small, so we neglect the $\mathbf{E}\sin 2\xi'$ term in the denominator. For deviations δ smaller than 20° we can set $\sin\delta = \delta$ and $\cos\delta = 1$, which is quite reasonable considering that most ships will be magnetically compensated to some extent. Then we expand the denominator into a Fourier series in multiples of $2\xi'$ and divide through ("a somewhat laborious process of expansions and substitutions," as Smith puts it) to get the infinite series

$$\delta = A + B\sin\xi' + C\cos\xi' + D\sin 2\xi' + E\cos 2\xi' + F\sin 3\xi' + G\cos 3\xi' + \ldots$$

which we truncate to

$$\delta = A + B\sin\xi' + C\cos\xi' + D\sin 2\xi' + E\cos 2\xi'$$

This is the *inexact equation* for the magnetic deviation, as it uses (non-bolded) inexact coefficients. For **B**, **C** and **D** being small quantities of the first order and **A** and **E** being small quantities of the second order, the inexact coefficients A, B, C, D and E are found to be

$$\begin{aligned}
&\delta = A + B\sin\xi' + C\cos\xi' + D\sin 2\xi' + E\cos 2\xi' \\
&A = \arcsin\mathbf{A} = \text{a constant for the ship} \\
&B = \arcsin\frac{\mathbf{B}}{1 + {}^{1}/_{2}\sin D} \\
&C = \arcsin\frac{\mathbf{C}}{1 - {}^{1}/_{2}\sin D} \\
&D = \arcsin\mathbf{D} = \text{a constant for the ship} \\
&E = \arcsin\mathbf{E} = \text{a constant for the ship} \\
&\mathbf{B} = \frac{1}{\lambda}\left(c\tan\theta + \frac{P}{H}\right) \\
&\mathbf{C} = \frac{1}{\lambda}\left(f\tan\theta + \frac{Q}{H}\right)
\end{aligned}$$

The magnetic deviation δ is now approximated in terms of the compass course ξ' along with the horizontal force H and dip θ of the Earth's magnetic field at the ship's location (as provided by tables of measurements).

Here A, D and E are constants since they are simply arcsines of constants. The values of A, D, E, λ, c, P, f and Q are deduced for a particular ship from measurements of magnetic deviation as the ship is swung to various azimuths in a location of known horizontal magnetic force and dip.

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not only erroneous in principle but dangerous in practice." I believe this is Smith's introduction to Scoresby's book that caused an immediate printed response by Airy that I read about. It can be found at *http://books.google.com/books?id=MGgDAAAAYAAJ&printsec=titlepage&source=gbs_summary_-r&cad=0#PRA2-PA287,M1*.

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