

Archibald Smith's Derivation of the Basic Equations of Magnetic Derivation

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An addendum to the essay

Magnetic Deviation: Comprehension, Compensation and Computation

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THE ADMIRALTY MANUAL FOR THE DEVIATIONS OF THE COMPASS presents in great detail Archibald Smith's derivations of the various equations for magnetic deviation. Here we will briefly review the derivation for the simplest case, that of a ship on an even keel. The derivation is given in a number of the cited references of this essay, and the presentation here is a concise merger of these.

Let the magnetic force of the Earth relative to the ship be X , Y and Z in the forward, starboard and downward directions, and the total magnetic force of the ship and Earth be X' , Y' and Z' . In 1824 Siméon Denis-Poisson gave the following general equations relating these quantities, in which the induced magnetic forces from soft iron are proportional to the Earth's magnetic forces:

$$X' = X + aX + bY + cZ + P \quad (1)$$

$$Y' = Y + dX + eY + fZ + Q \quad (2)$$

$$Z' = Z + gX + hY + kZ + R \quad (3)$$

Archibald Smith added the constant terms (P , Q and R) to account for the ship's permanent magnetism due to hard iron, as Poisson considered these effects to be scattered without a significant overall impact, which was true for wooden ships of the time. Poisson also assumed certain symmetries in the iron of the ship that Smith did not.

Now assume that the ship's head is pointing an angle ξ from magnetic north at its location (the *magnetic course*), and an angle ξ' as shown on the compass (the *compass course*). The magnetic deviation δ of the compass needle from magnetic north due to the ship is then $\xi - \xi'$.

For this derivation,

$$H = \text{the horizontal force of the magnetic field of the Earth} = \sqrt{X^2 + Y^2}$$

$$H' = \text{the combined horizontal force of the magnetic field of the Earth and ship} = \sqrt{X'^2 + Y'^2}$$

$$\theta = \text{the dip, or inclination, of the magnetic field of the Earth}$$

The vertical force of the Earth's magnetic field Z does not appear explicitly in this derivation as it can be

represented by $H \tan \theta$. It is apparent that

$$\begin{aligned} X &= H \cos \xi \\ Y &= -H \sin \xi \\ Z &= H \tan \theta \\ X' &= H' \cos \xi \\ Y' &= -H' \sin \xi \end{aligned}$$

and substituting these into Equations (1) and (2) we have

$$H' \cos \xi = (1+a)H \cos \xi - bH \sin \xi + cH \tan \theta + P \quad (4)$$

$$-H' \sin \xi = (-\sin \xi + d \cos \xi)H - eH \sin \xi + fH \tan \theta + Q \quad (5)$$

Let's multiply the first of these by $\sin \xi$ and the second by $\cos \xi$ and add them. After some simplification we arrive at

$$\frac{H'}{H} \sin \delta = \frac{d-b}{2} + \left(c \tan \theta + \frac{P}{H} \right) \sin \xi + \left(f \tan \theta + \frac{Q}{H} \right) \cos \xi + \frac{a-e}{2} \sin 2\xi + \frac{b+d}{2} \cos 2\xi \quad (6)$$

This provides the directive force to magnetic east due to the ship in units of the Earth's horizontal force, and it has a mean value of $(d-b)/2$. The following identities were used to obtain this form of the equation:

$$\begin{aligned} \cos^2 \xi &= \frac{1 + \cos 2\xi}{2} \\ \sin^2 \xi &= \frac{1 - \cos 2\xi}{2} \\ \sin \xi \cos \xi &= \frac{\sin 2\xi}{2} \end{aligned}$$

Now we multiply Equation (4) by $\cos \xi$ and Equation (5) by $\sin \xi$ and subtract them. After a similar simplification we arrive at

$$\frac{H'}{H} \cos \delta = 1 + \frac{a+e}{2} + \left(c \tan \theta + \frac{P}{H} \right) \cos \xi - \left(f \tan \theta + \frac{Q}{H} \right) \sin \xi + \frac{a-e}{2} \cos 2\xi - \frac{b+d}{2} \sin 2\xi \quad (7)$$

This provides the directive force to magnetic north due to the Earth and ship in units of the Earth's horizontal force, and it has a mean value of $1 + (a+e)/2$, which we will denote as λ , so λH is the mean force to magnetic north.

Let's define the following constants:

$$\begin{aligned} \mathbf{A} &= \frac{d-b}{2\lambda} \\ \mathbf{B} &= \frac{1}{\lambda} \left(c \tan \theta + \frac{P}{H} \right) \\ \mathbf{C} &= \frac{1}{\lambda} \left(f \tan \theta + \frac{Q}{H} \right) \\ \mathbf{D} &= \frac{a-e}{2\lambda} \\ \mathbf{E} &= \frac{b+d}{2\lambda} \end{aligned}$$

These constants are in bold font here, but historically they are almost always printed in German (Blackletter) font as the hard-to-distinguish \mathfrak{A} , \mathfrak{B} , \mathfrak{C} , \mathfrak{D} and \mathfrak{E} . If we divide Equations (6) by (7) we get

$$\tan \delta = \frac{\mathbf{A} + \mathbf{B} \sin \xi + \mathbf{C} \cos \xi + \mathbf{D} \sin 2\xi + \mathbf{E} \cos 2\xi}{1 + \mathbf{B} \cos \xi - \mathbf{C} \sin \xi + \mathbf{D} \cos 2\xi - \mathbf{E} \sin 2\xi} \quad (8)$$

This is the *exact equation* for the deviation δ in terms of the magnetic course ξ , constants \mathbf{A} , \mathbf{D} and \mathbf{E} that are dependent only on the arrangement of soft and hard iron in the particular ship, and constants \mathbf{B} and \mathbf{C} that are dependent on the Earth's magnetic field at the ship's location as well as the arrangement of its soft and hard iron.

But at sea we really don't know the magnetic course ξ of the ship's head, just the compass course ξ' that we are reading. This is a real problem, because even when we substitute $\xi - \xi'$ for δ we don't have a closed form solution for finding ξ (and therefore δ). It's much better to find a way to express the deviation δ directly in terms of the compass course ξ' . In the process of doing this, we will end up replacing the exact coefficients \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{D} and \mathbf{E} with inexact coefficients A , B , C , D and E .

We can do this in the following manner. First, we substitute $\xi' + \delta$ for ξ and $\sin \delta / \cos \delta$ for $\tan \delta$ in Equation (8). After some simplification we get

$$\sin \delta = \mathbf{A} \cos \delta + \mathbf{B} \sin \xi' + \mathbf{C} \cos \xi' + \mathbf{D} \sin(2\xi' + \delta) + \mathbf{E} \cos(2\xi' + \delta)$$

which can also be expressed as

$$\sin \delta = \frac{\mathbf{A} \cos \delta + \mathbf{B} \sin \xi' + \mathbf{C} \cos \xi' + \mathbf{D} \sin 2\xi' + \mathbf{E} \cos 2\xi'}{1 - \mathbf{D} \cos 2\xi' + \mathbf{E} \sin 2\xi'}$$

It turns out that \mathbf{E} is relatively small, so we neglect the $\mathbf{E} \sin 2\xi'$ term in the denominator. For deviations δ smaller than 20° we can set $\sin \delta = \delta$ and $\cos \delta = 1$, which is quite reasonable considering that most ships will be magnetically compensated to some extent. Then we expand the denominator into a Fourier series in multiples of $2\xi'$ and divide through ("a somewhat laborious process of expansions and substitutions," as Smith puts it) to get the infinite series

$$\delta = A + B \sin \xi' + C \cos \xi' + D \sin 2\xi' + E \cos 2\xi' + F \sin 3\xi' + G \cos 3\xi' + \dots$$

which we truncate to

$$\delta = A + B \sin \xi' + C \cos \xi' + D \sin 2\xi' + E \cos 2\xi'$$

This is the *inexact equation* for the magnetic deviation, as it uses (non-bolded) inexact coefficients. For \mathbf{B} , \mathbf{C} and \mathbf{D} being small quantities of the first order and \mathbf{A} and \mathbf{E} being small quantities of the second order, the

inexact coefficients A, B, C, D and E are found to be

$$\delta = A + B \sin \xi' + C \cos \xi' + D \sin 2\xi' + E \cos 2\xi'$$

$$A = \arcsin \mathbf{A} = \text{a constant for the ship}$$

$$B = \arcsin \frac{\mathbf{B}}{1 + 1/2 \sin D}$$

$$C = \arcsin \frac{\mathbf{C}}{1 - 1/2 \sin D}$$

$$D = \arcsin \mathbf{D} = \text{a constant for the ship}$$

$$E = \arcsin \mathbf{E} = \text{a constant for the ship}$$

$$\mathbf{B} = \frac{1}{\lambda} \left(c \tan \theta + \frac{P}{H} \right)$$

$$\mathbf{C} = \frac{1}{\lambda} \left(f \tan \theta + \frac{Q}{H} \right)$$

The magnetic deviation δ is now approximated in terms of the compass course ξ' along with the horizontal force H and dip θ of the Earth's magnetic field at the ship's location (as provided by tables of measurements). Here A, D and E are constants since they are simply arcsines of constants. The values of $A, D, E, \lambda, c, P, f$ and Q are deduced for a particular ship from measurements of magnetic deviation as the ship is swung to various azimuths in a location of known horizontal magnetic force and dip.